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PERIODIC ORBITS IN TRIGONOMETRIC SERIES

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ABSTRACT

A method is given for the study of families of periodic motion using trigonometric series to represent the individual solutions. The continuous deformations of a periodic orbit along a family are represented by the variations of the trigonometric coefficients with respect to a parameter.

For the applications the series are truncated, while the coefficients and their variations are determined numerically. In this form, the continuation with respect to the parameter is given by a mapping f: $R^n \to R^n$ of the space of coefficients into itself. The values of the coefficients are then improved by another mapping which is a contraction operator in some neighborhood of the fixed point representing the solution.

The method is applied to the natural families defined by Wintner and to the families of the first and second kinds of Poincaré in the restricted problem of three bodies.

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CONTENTS

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INTRODUCTION

Trigonometric series have been used very extensively in the study of periodic orbits and families of periodic motion in the restricted problem of three bodies. Many of these studies depend on the presence of small parameters in the problem, and the results consist of trigonometric terms whose coefficients are power series in the small parameters. That is to say, the results are obtained in the form of Poisson series. The same techniques can be applied using numerical values for the parameters, and this is referred to as a semi-analytic development. However, many families of periodic orbits have not yielded to these methods and have thus far been explored only by numerical integration.

The purpose of the present study is to obtain semi-analytic results for some of these more difficult families of periodic orbits and to place the continuation of these families on a sound basis when trigonometric series with numerical coefficients are used in a method of successive approximations.

Cases of near and exact resonance are of special interest and are handled without difficulty using the present technique. The periodic matrix of the variational equations plays an essential role in the method, and the linear stability analysis is done in the usual way.

A good discussion of the method, as applied to isolated periodic solutions has been given by Urabe (1965). Numerical examples are given by Urabe and Reiter (1966). More recent results and references are given by Stokes (1969). The successive approximations are similar to those of Bennett and Palmore (1968) in that each step yields a periodic function which is an approximate solution of the original equations of motion. The variations of the series coefficients with re ect to a local parameter give a representation of the functions $x_1(t)$ and $y_1(t)$ discussed by Deprit and Henrard (1967) (page 160).

The method described in this study has been applied in the computation of periodic orbits in the restricted problem of four bodies by Kolenkiewicz and Carpenter (1967 and 1968) and in the restricted problem of three bodies by Carpenter and Stumpff (1968). Further applications are being made by Deprit and Carpenter for locating elements of many new families or orbits in the problem of three bodies.

THE ITERATION PROCESS FOR FIXED PERIOD

The method is applicable to a wide range of problems, and a general formulation is possible. However, each class of problems has its own interesting features, and it is usually possible to greatly improve the efficiency by taking advantage of the particular forms of the equations and the solutions.

The formulation will be given for the restricted problem of three-bodies considering symmetric orbits which lie in the plane of motion of the primaries. Furthermore, the method will be developed as a modification of an iterative general perturbations technique, so a two-body reference orbit is used as the starting point. For convenience the reference orbit is circular with respect to the central primary whose mass is put equal to M (referred to as the Sun).

Let a and n be the radius and mean motion respectively of the circular reference orbit of the infinitesimal particle (referred to as the minor planet), and let k be the Gaussian constant so that

$$n^2 a^3 = k^2 M .$$

Let the true position of the minor planet be given by

$$\mathbf{r} = (1 + \alpha)\, \mathbf{r}_0 + \beta \mathbf{w}$$

where $\mathbf{r}_0$ is the position in the reference orbit,

$$\mathbf{w} = \frac{1}{n} \frac{d\mathbf{r}_0}{dt}$$

is the vector of length a in the direction of the velocity in the reference orbit, and the quantities α and β will represent the periodic deviations from the circular motion.

Let a' and n' be the radius and mean motion respectively of the circular motion of Jupiter, whose mass is denoted by m, so that

$$n'^2 a'^3 = k^2 (M + m) \quad \text{or} \quad n'^2 a'^3 = k^2 M (1 + m') \quad \text{with} \quad m' = m/M$$

and

$$a'/a = (1 + m')^{1/3}\, \nu^{2/3}$$

where

$$\nu = n/n'.$$

For the cases where $n > n'$, the relative reference motions will be periodic with period

$$T_r = \frac{2\pi}{n - n'}$$

and the orbit to be determined will have the period

$$T = \frac{2\pi}{N}$$

with

$$N = \frac{n - n'}{\ell}$$

for some positive integer value of ℓ. The integer ℓ is the winding number or index of the orbit with respect to the Sun in the synodic coordinate system. Thus the trigonometric argument is

$$\theta = Nt.$$

For symmetric orbits the epoch of time is chosen such that the minor planet crosses the Sun-Jupiter line at $t = 0$.

With respect to the mean anomaly, $g = nt$, the equations of motion for the minor planet are

$$\frac{d^2\alpha}{dg^2} - 2\frac{d\beta}{dg} - 3\alpha = \Omega_\alpha$$

$$\frac{d^2\beta}{dg^2} + 2\frac{d\alpha}{dg} = \Omega_\beta.$$

In terms of the present coordinates

$$\Omega = \frac{1}{2}\left(\frac{r}{a}\right)^2 + \frac{a}{r} - \frac{3}{2}\alpha^2 + m'\left\{\frac{a}{\rho} - \left(\frac{a}{a'}\right)^2 [(1+\alpha)\cos \ell\theta - \beta \sin \ell\theta]\right\}$$

where

$$\frac{r}{a} = \sqrt{(1+\alpha)^2 + \beta^2}$$

and

$$\frac{a}{\rho} = \left\{(1+\alpha)^2 + \beta^2 + \left(\frac{a'}{a}\right)^2 - 2\left(\frac{a'}{a}\right)[(1+\alpha)\cos \ell\theta - \beta \sin \ell\theta]\right\}^{-1/2},$$

ρ being the distance from the minor planet to Jupiter. Thus

$$\Omega_\alpha = (1+\alpha)\left[1 - \left(\frac{a}{r}\right)^3\right] - 3\alpha$$

$$+ m'\left\{\left(\frac{a}{\rho}\right)^3\left[\left(\frac{a'}{a}\right)\cos \ell\theta - (1+\alpha)\right] - \left(\frac{a}{a'}\right)^2 \cos \ell\theta\right\}$$

and

$$\Omega_\beta = \beta\left[1 - \left(\frac{a}{r}\right)^3\right] + m'\left\{-\left(\frac{a}{\rho}\right)^3\left[\left(\frac{a'}{a}\right)\sin \ell\theta + \beta\right] + \left(\frac{a}{a'}\right)^2 \sin \ell\theta\right\}.$$

Later we will also need expressions for the partial derivatives of Ω_α and Ω_β with respect to α, β, m' and ν. These are

$$\Omega_{\alpha\alpha} = -2 - \left(\frac{a}{r}\right)^3 + 3(1+\alpha)^2\left(\frac{a}{r}\right)^5 + m'\left\{-\left(\frac{a}{\rho}\right)^3 + 3\left(\frac{a}{\rho}\right)^5\left[\left(\frac{a'}{a}\right)\cos \ell\theta - (1+\alpha)\right]^2\right\}$$

$$\Omega_{\alpha\beta} = \Omega_{\beta\alpha} = 3(1+\alpha)\beta\left(\frac{a}{r}\right)^5 - 3m'\left(\frac{a}{\rho}\right)^5\left[\left(\frac{a'}{a}\right)\cos \ell\theta - (1+\alpha)\right]\left[\left(\frac{a'}{a}\right)\sin \ell\theta + \beta\right]$$

$$\Omega_{\beta\beta} = 1 - \left(\frac{a}{r}\right)^3 + 3\beta^2\left(\frac{a}{r}\right)^5 + m'\left\{-\left(\frac{a}{\rho}\right)^3 + 3\left(\frac{a}{\rho}\right)^5\left[\left(\frac{a'}{a}\right)\sin \ell\theta + \beta\right]^2\right\}$$

$$\Omega_{\alpha\nu} = \frac{2}{3}\,\frac{a'}{a}\,\frac{m'}{\nu}\left\{3\left(\frac{a}{\rho}\right)^5\left[(1+\alpha)\cos\ell\theta - \beta\sin\ell\theta - \left(\frac{a'}{a}\right)\right]\left[\left(\frac{a'}{a}\right)\cos\ell\theta - (1+\alpha)\right]\right.$$

$$\left. + \left[\left(\frac{a}{\rho}\right)^3 + 2\left(\frac{a}{a'}\right)^3\right]\cos\ell\theta\right\}$$

$$\Omega_{\beta\nu} = -\frac{2}{3}\,\frac{a'}{a}\,\frac{m'}{\nu}\left\{3\left(\frac{a}{\rho}\right)^5\left[(1+\alpha)\cos\ell\theta - \beta\sin\ell\theta - \left(\frac{a'}{a}\right)\right]\left[\left(\frac{a'}{a}\right)\sin\ell\theta + \beta\right]\right.$$

$$\left. + \left[\left(\frac{a}{\rho}\right)^3 + 2\left(\frac{a}{a'}\right)^3\right]\sin\ell\theta\right\}$$

$$\Omega_{\alpha m'} = \frac{1}{2}\,\frac{\nu}{1+m'}\,\Omega_{\alpha\nu} + \left\{\left(\frac{a}{\rho}\right)^3\left[\left(\frac{a'}{a}\right)\cos\ell\theta - (1+\alpha)\right] - \left(\frac{a}{a'}\right)^2\cos\ell\theta\right\}$$

$$\Omega_{\beta m'} = \frac{1}{2}\,\frac{\nu}{1+m'}\,\Omega_{\beta\nu} - \left\{\left(\frac{a}{\rho}\right)^3\left[\left(\frac{a'}{a}\right)\sin\ell\theta + \beta\right] - \left(\frac{a}{a'}\right)^2\sin\ell\theta\right\}$$

The dependence on ν and part of the dependence on m' comes from the relation

$$\frac{a'}{a} = (1+m')^{1/3}\,\nu^{2/3}\,.$$

The argument, θ, is treated as an independent quantity, because these expressions are used for computing the variations of the coefficients in the series expansions.

For the symmetric periodic orbits, α and Ω_α are even functions of θ while β and Ω_β are odd so that

$$\alpha = \sum_{k=0}^{\infty} \alpha_k \cos k\,\theta$$

$$\beta = \sum_{k=1}^{\infty} \beta_k \sin k\,\theta$$

$$\Omega_\alpha = \sum_{k=0}^{\infty} c_k \cos k\,\theta$$

$$\Omega_\beta = \sum_{k=1}^{\infty} s_k \sin k\,\theta$$

for non-collision orbits. All such series are truncated in the actual computations.

Putting

$$N_k = k\,\frac{N}{n}$$

and substituting the series expressions into the equations of motion gives

$$\left.\begin{aligned} (3 + N_k^2)\,\alpha_k + 2N_k\,\beta_k &= -\,c_k \\ 2N_k\,\alpha_k + N_k^2\,\beta_k &= -\,s_k \end{aligned}\right\}\quad k = 0,\ 1,\ 2,\ \ldots \qquad \text{(A)}$$

by equating coefficients. If c_k and s_k were known, the solution would be obtained from the formulas

$$\alpha_0 = -c_0/3$$

$$\left.\begin{aligned} \alpha_k &= -\frac{1}{N_k^2 - 1} c_k + \frac{2}{N_k(N_k^2 - 1)} s_k \\ \beta_k &= \frac{2}{N_k(N_k^2 - 1)} c_k - \frac{3 + N_k^2}{N_k^2(N_k^2 - 1)} s_k \end{aligned}\right\} \quad k = 1, 2, 3, \ldots . \tag{B}$$

Since Ω_α and Ω_β depend on α and β, the coefficients c_k and s_k are not known beforehand. However, in many practical cases the solution may be obtained by the iterative general perturbations technique. This consists of starting with $\alpha = \beta = 0$, for example, computing approximate values for c_k and s_k by harmonic analysis of the expressions for Ω_α and Ω_β, and using these values to solve for approximate values of α_k and β_k. These approximations for α and β are then used in a new harmonic analysis of Ω_α and Ω_β to give the next approximations for α_k and β_k etc.

This simple process fails in some very interesting cases such as resonance. For example, when

$$\frac{n}{n'} = \frac{p}{q}$$

for integer values of p and q, and

$$\ell = p - q$$

the formula gives

$$N_p = 1 .$$

In this case the formulas for α_k and β_k have zero divisors at $k = p$, so the method doesn't work, although the original equations could have a solution with $c_p = 2 s_p$.

The simple iteration process has been modified to handle cases such as the one just described by taking into account the partial derivatives of c_k and s_k with respect to α_j and β_j and then computing corrections to an approximate solution. This modified process yielded, for example, a linearly stable orbit for $n/n' = 3/2$ (the Hilda resonance). This particular orbit will be discussed later.

The partial derivatives of c_k and s_k with respect to α_j and β_j are obtained from the coefficients in the expansions of $\Omega_{\alpha\alpha}$, $\Omega_{\alpha\beta}$ and $\Omega_{\beta\beta}$. These functions can be represented in the form

$$\Omega_{\alpha\alpha} = \sum_{k=0}^{\infty} x_k \cos k\theta$$

$$\Omega_{\alpha\beta} = \sum_{k=1}^{\infty} y_k \sin k\theta$$

$$\Omega_{\beta\beta} = \sum_{k=0}^{\infty} z_k \cos k\theta$$

Numerical values of the coefficients are computed by harmonic analysis. Then

$$\frac{\partial c_k}{\partial \alpha_j} = \frac{x_{|k-j|} + x_{k+j}}{2}$$

$$\frac{\partial c_k}{\partial \beta_j} = \frac{-\eta_{k,j}\, y_{|k-j|} + y_{k+j}}{2}$$

$$\frac{\partial s_k}{\partial \alpha_j} = \frac{\eta_{k,j}\, y_{|k-j|} + y_{k+j}}{2}$$

$$\frac{\partial s_k}{\partial \beta_j} = \frac{z_{|k-j|} - z_{k+j}}{2}$$

where

$$\eta_{k,j} = \begin{cases} 1 & \text{for } k \geq j \\ -1 & \text{for } k < j . \end{cases}$$

These variations are the quantities needed for improving the iteration formulas which can now be written as

$$\left.\begin{aligned} (3 + N_k^2)\,\delta\alpha_k + 2N_k\,\delta\beta_k + \sum_{i=0}^{\infty} \frac{\partial c_k}{\partial \alpha_i}\,\delta\alpha_i + \sum_{i=1}^{\infty} \frac{\partial c_k}{\partial \beta_i}\,\delta\beta_i &= \epsilon_k \\ 2N_k\,\delta\alpha_k + N_k^2\,\delta\beta_k + \sum_{i=0}^{\infty} \frac{\partial s_k}{\partial \alpha_i}\,\delta\alpha_i + \sum_{i=1}^{\infty} \frac{\partial s_k}{\partial \beta_i}\,\delta\beta_i &= \delta_k \end{aligned}\right\}\ k=0,1,2,\ldots \qquad \text{(C)}$$

where

$$\left.\begin{aligned} \epsilon_k &= -(3 + N_k^2)\,\alpha_k - 2\,N_k\,\beta_k - c_k \\ \delta_k &= -2\,N_k\,\alpha_k - N_k^2\,\beta_k - s_k \end{aligned}\right\}\ k = 0,\ 1,\ 2,\ \ldots$$

The quantities $\delta\alpha_k$ and $\delta\beta_k$ to be computed are corrections to the approximate values α_k and β_k.

In the applications the series appearing in this system of equations can be truncated at a relatively low order depending on the rate of convergence for the particular orbit being computed. The remaining equations are then uncoupled in pairs as before.

One must also consider the possibility that the symmetric matrix associated with this system of linear equations may be singular. The places where this has occurred in the applications is at those points on a natural family of orbits where the period becomes stationary. These cases occur near a resonance and are easily treated by holding the eccentricity fixed rather than the period as in the following discussion.

THE ECCENTRICITY OF THE ORBITS

In many cases, such as resonance, the eccentricity is a convenient parameter to be used in following a family of orbits. For two-body motion the position vector may be written as

$$\mathbf{r} = a\,\mathbf{P}(\cos E - e) + a\sqrt{1 - e^2}\,\mathbf{Q}\sin E$$

where $\mathbf{P}$ and $\mathbf{Q}$ are the usual unit vectors, a is the semi-major axis, e is the eccentricity, and E is the eccentric anomaly. The deviations from circular motion may be expressed in α and β by considering

$$\mathbf{r} = (1 + \alpha)\,\mathbf{r}_0 + \beta\,\mathbf{w}$$

where

$$\mathbf{r}_0 = a\mathbf{P}\cos g + a\,\mathbf{Q}\sin g$$

$$\mathbf{w} = -\,a\mathbf{P}\sin g + a\,\mathbf{Q}\cos g$$

with g as the mean anomaly. Equating the coefficients of $\mathbf{P}$ in the two expressions for $\mathbf{r}$, it follows that

$$\cos E - e = (1 + \alpha)\cos g - \beta\sin g.$$

When cos E is expanded in a cosine series in g, the constant term is $-e/2$. Now

$$\alpha = \sum_{k=0}^{\infty} \alpha_k \cos k\,\theta$$

$$\beta = \sum_{k=1}^{\infty} \beta_k \sin k\,\theta,$$

and, for the motion to be periodic in the rotating system

$$\theta = g/p$$

for some positive integer p. Therefore

$$e = (\beta_p - \alpha_p)/3.$$

This two-body formula may be used to define the mean eccentricity of a perturbed orbit. For elliptic motion all the coefficients in the expansions of α and β can be computed from e using the Bessel functions, and in many cases this is a good initial approximation for the perturbed periodic orbit.

The eccentricity may now be prescribed, and this eliminates one equation in the system by putting

$$\beta_p = 3\,e_p + \alpha_p .$$

The subscript is placed on e to identify the coefficients with which it is associated. The equation which has been eliminated is replaced by another for determining the period or the mass ratio as in the following section.

THE GENERAL PREDICTOR-CORRECTOR FORMULAS

The Equations (A) relating the coefficients can be written in matrix form as

$$DX = f(X)$$

where X is the vector of eoefficients α_k and β_k , f(X) is the vector of coefficients $-c_k$ and $-s_k$ which were seen to depend on X, and D is the matrix of coefficients multiplying α_k and β_k in Equations (A). It is assumed that the sequence of equations is truncated at some appropriate value of k. Note that the matrix D depends only on the ratio, $\nu = n/n'$, of mean motions for fixed ℓ. However, f depends on m′ as well as ν. Allowing first order corrections, the equations become

$$\left(D - \frac{\partial f}{\partial X}\right) \delta X + \left(\frac{\partial D}{\partial \nu} X - \frac{\partial f}{\partial \nu}\right) \delta \nu - \frac{\partial f}{\partial m'} \delta m' = f - DX.$$

The components of $\partial f / \partial \nu$ and $\partial f / \partial m'$ are simply the coefficients in the trigonometric expansions of $\Omega_{\alpha\nu}$, $\Omega_{\beta\nu}$, $\Omega_{\alpha m'}$ and $\Omega_{\beta m'}$. With ν and m' fixed, the iteration (or corrector) formulas become

$$\delta X = \left(D - \frac{\partial f}{\partial X}\right)^{-1} (f - DX). \tag{D}$$

From an established orbit, the predictor formulas which give the variations of X with respect to the parameters are

$$\frac{\partial X}{\partial \nu} = \left(D - \frac{\partial f}{\partial X}\right)^{-1} \left(\frac{\partial f}{\partial \nu} - \frac{\partial D}{\partial \nu} X\right) \tag{E}$$

and

$$\frac{\partial X}{\partial m'} = \left(D - \frac{\partial f}{\partial X}\right)^{-1} \frac{\partial f}{\partial m'} \tag{F}$$

so that the same symmetric matrix is to be inverted in each case.

The eccentricity, e_p , is introduced as a parameter using the relation

$$\delta \beta_p = 3\, \delta e_p + \delta \alpha_p$$

wherever $\delta \beta_p$ occurs in the equations. Then, when δe_p is specified, either $\delta \nu$ or $\delta m'$ is obtained as part of the solution. This is accomplished by simple manipulations of the vectors and matrices in the above formulas. After the solution, the new value of β_p is computed from

$$\beta_p = 3\, e_p + \alpha_p$$

THE LINEAR STABILITY ANALYSIS

Putting

$$\gamma = d\alpha/dg$$

and

$$\delta = d\beta/dg$$

the equations of motion can be written as

$$d\alpha/dg = \gamma$$

$$d\beta/dg = \delta$$

$$d\gamma/dg = 3\alpha + 2\delta + \Omega_\alpha$$

$$d\delta/dg = -2\gamma + \Omega_\beta$$

so that the variational equations are

$$d\Phi/dg = A\Phi$$

with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3+\Omega_{\alpha\alpha} & \Omega_{\alpha\beta} & 0 & 2 \\ \Omega_{\alpha\beta} & \Omega_{\beta\beta} & -2 & 0 \end{bmatrix}$$

The functions $\Omega_{\alpha\alpha}$, $\Omega_{\alpha\beta}$ and $\Omega_{\beta\beta}$ are available in series form from the iteration process, so the variational equations can be integrated directly. For the present this is being done numerically using power series.

THE APPLICATIONS

Previous applications were mentioned in the introduction. Some additional orbits and families of orbits are given here as illustrations. These orbits all close after one revolution around the sun in the synodic system ($\ell = 1$ in the definition of the trigonometric argument).

A typical stable orbit of the first kind is shown in the synodic coordinate system in Figure 1a. Such orbits are nearly circular and very easy to compute. The deviations from the circular reference motion are shown in the α, β plot of Figure 1b, and the series coefficients are given in Table 1. The values are given to six decimals for purposes of illustration. The orbits were all computed to an accuracy of about 12 decimals.

As the ratio of mean motions, $\nu = n/n'$, is decreased toward the resonance value, $\nu = 2$, the deviations from circular motion increase as shown in Figure 2a for the stable orbit at $\nu = 2.001$. The α, β plot of Figure 2b for this orbit consists of two loops which are nearly identical. The series coefficients, given in Table 2, are much larger now, but the rate of convergence is still good. This orbit is best described as a slowly rotating perturbed ellipse with eccentricity $e_2 = 0.388739$. The coefficients with odd subscripts would all be zero if the orbit were a true ellipse.

There are more periodic orbits which are nearly circular for $3/2 < \nu < 2/1$. The deviations from circular motion for one such orbit are shown in Figure 3 where $\nu = 1.6$. This orbit is also stable and has the series coefficients shown in Table 3.

As the ratio of mean motions is reduced toward the 3/2 resonance value, the eccentricity increases again. There is a stable orbit at the exact resonance shown in Figure 4a. Now the α, β plot, Figure 4b, consists of three nearly identical loops. The dominant coefficients (with subscripts which are multiples of 3) are given in Table 4. This orbit is a slightly perturbed ellipse with $e_3 = 0.453692$ and no secular motion of the perihelion.

Each of the above orbits was computed as a member of a natural family with $m' = 1/1047.35$ and using $\nu = n/n'$ as the parameter. For the orbit of Figure 4 this would not have been possible without using the partial derivatives in the iteration because of the zero divisors. For the orbits near the resonances, $\nu = 2/1$ and $\nu = 3/2$, the eccentricity, e_2 or e_3, would serve as a better parameter.

Each of the orbits except the one of Figure 4 has also been computed starting from circular two-body motion ($m' = 0$) and then increasing m' to the value for Jupiter while holding ν fixed. This is the technique for orbits of the first kind (Poincaré). The variations of e_2 with m' for three families of this type are shown in Figure 5.

Each of the orbits has also been computed by starting from elliptic two-body motion ($m' = 0$) and then increasing m' to the value for Jupiter while holding the eccentricity e_p fixed as in the method for orbits of the second kind (Poincaré).

In this approach $p = 2$ for the orbits of Figures 1 and 2 while $p = 3$ for the orbit of Figure 4. As a numerical experiment the orbit of Figure 3 was computed by continuation from two different two-body elliptic orbits, one starting from $\nu = 2$ and holding e_2 fixed, and the other starting from $\nu = 3/2$ and holding e_3 fixed. The variations of ν with respect to m' for these two families along with three others are shown in Figure 6.

In following families of the types illustrated in Figures 5 and 6 there will be cases where the curves have vertical tangents (the mass m' is stationary at a point on the family). The mass m' is not a suitable parameter in a neighborhood of such a point, and the matrix involved in the computation becomes singular. In such cases the ratio of mean motions, ν, and the eccentricity, e_p, are specified, while m' is obtained in the solution along with the other coefficients. This is somewhat akin to prescribing a function and then searching for the problem which it solves, but in this case the procedure is necessary and justified for the purpose of continuing a family of orbits.

The relationship between the α, β and synodic coordinate systems is indicated in Figure 7.

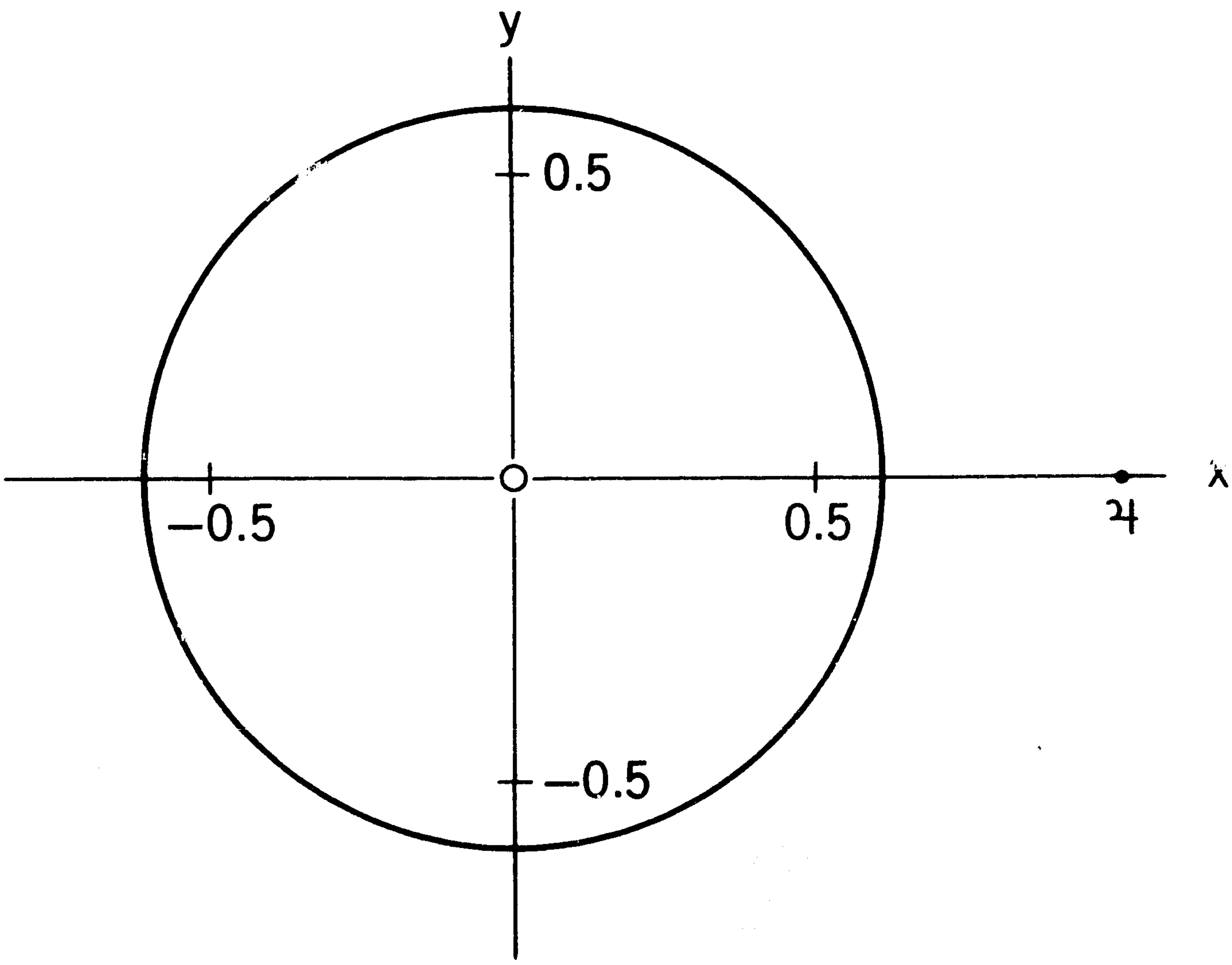

Figure 1a–Nearly circular stable periodic orbit of the first hand with mean motion slightly greater than that of Hecuba resonance. $m' = 1/1047.35$, $\nu = n/n' = 2.1$, $e_2 = 0.011905$. (Synodic coordinate system with the sun at the origin and Jupiter at $X = 1$.)

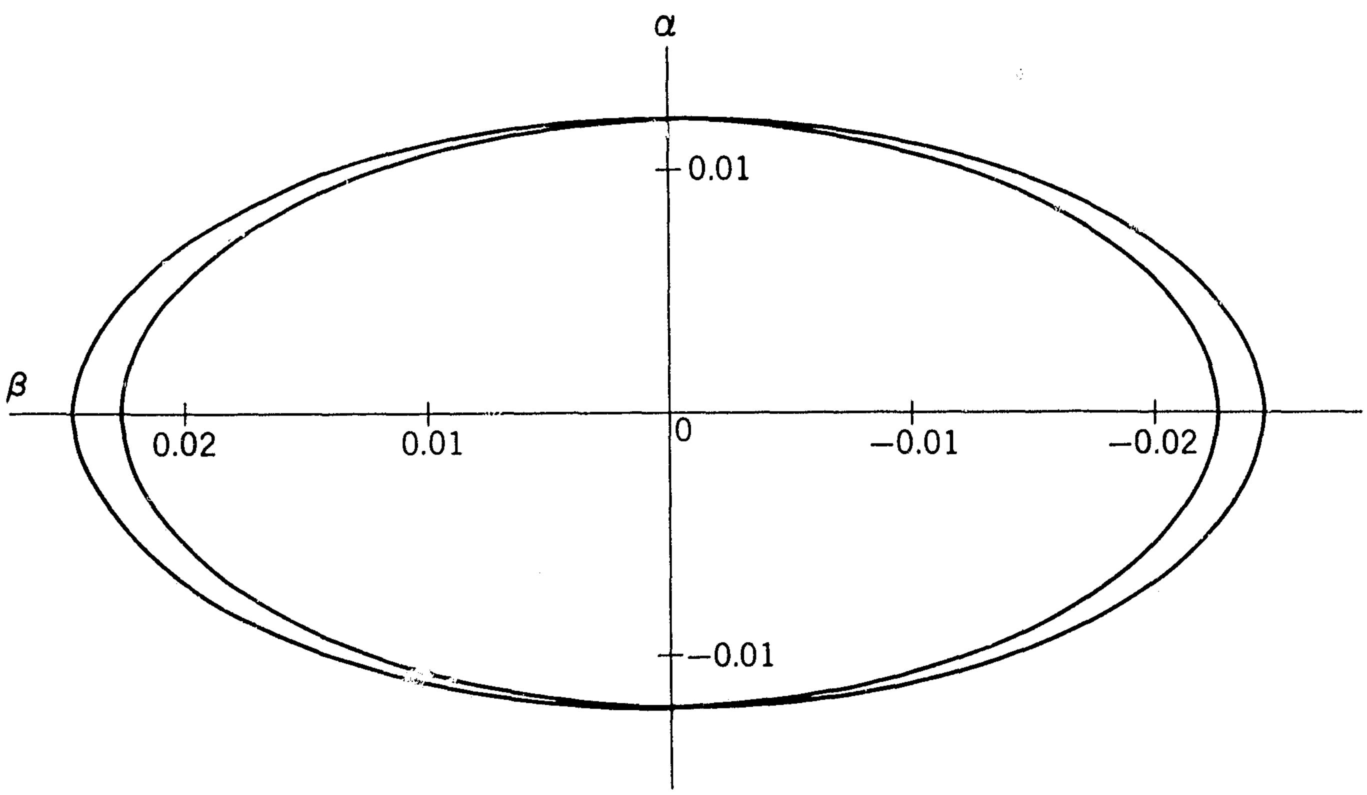

Figure 1b—Deviations from circular motion for the orbit of Figure 1a. The two loops cross at the top of the figure, while at the bottom they are nearly tangent but do not cross. Uniform circular motion would give $\alpha = \beta = 0$ throughout the orbit. The sun is at $\alpha = -1$ while Jupiter moves on the circle $(1 + \alpha)^2 + \beta^2 = (a'/a)^2$.

Table 1
Coefficients of the trigonometric series for α and β in the orbit of Figures 1a and 1b

k	$\alpha_k \cdot 10^6$	$\beta_k \cdot 10^6$
0	-122	0
1	595	-2028
2	-12158	23556
3	-491	719
4	-60	202
5	-43	57
6	-18	23
7	-8	10
8	-4	4
9	-2	2
10	-1	1
11	-1	1

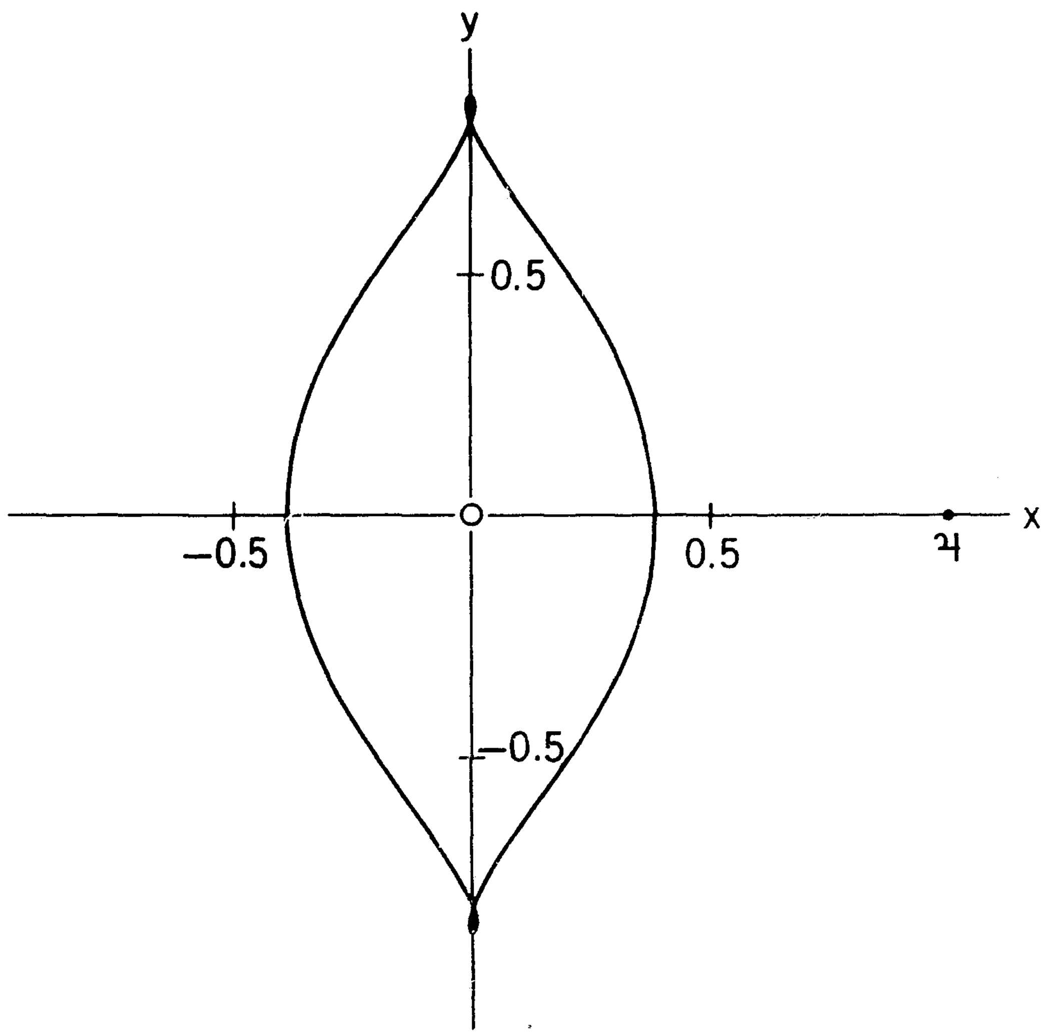

Figure 2a–Synodic motion of a stable periodic orbit with mean motion very near resonance. $m' = 1/1047.35$, $\nu = 2.001$, $e_2 = 0.388\,739$.

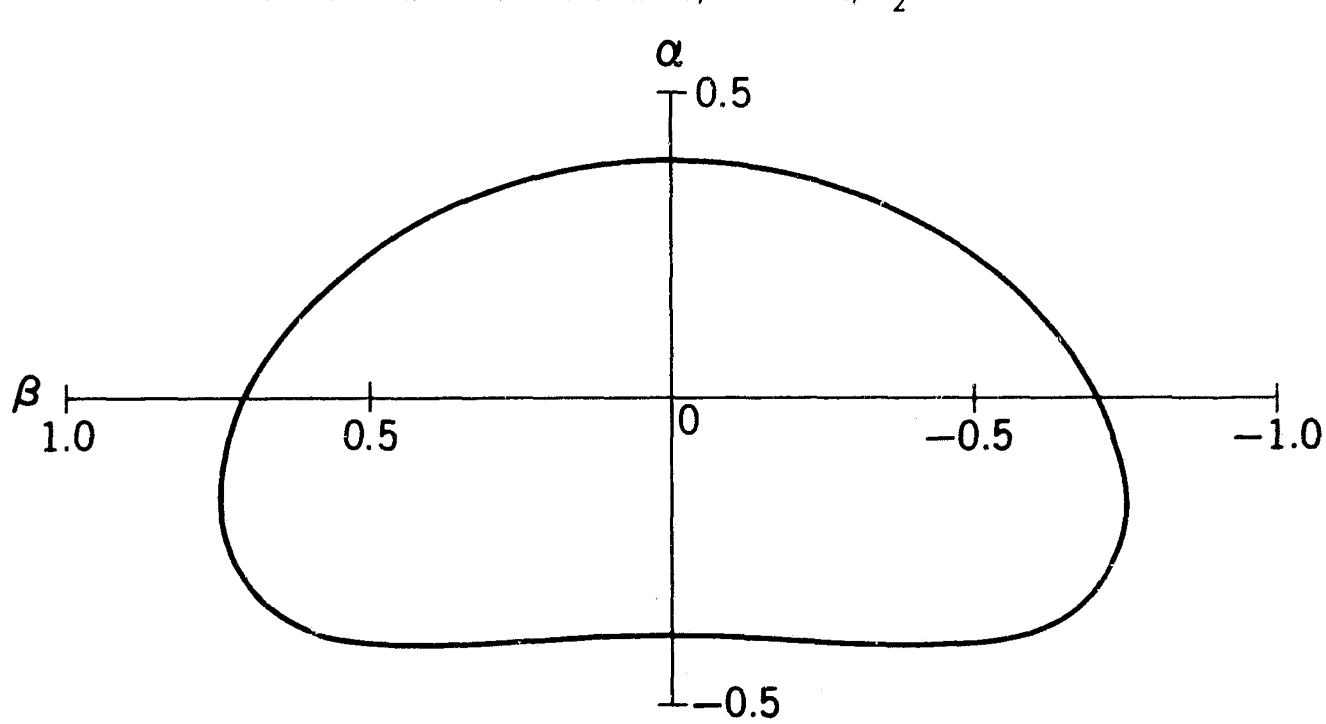

Figure 2b–Deviations from circular motion for the orbit of Figure 2a. The figure consists of two nearly identical loops. The shape is typical of elliptic deviations from circular motion.

Table 2
Coefficients of the trigonometric series
for the orbit of Figures 2a and 2b

k	$\alpha_k \cdot 10^6$	$\beta_k \cdot 10^6$
0	-75926	0
1	-1433	2179
2	-410408	755808
3	+198	334
4	68305	28584
5	+149	128
6	18652	13566
7	+72	65
8	6429	5342
9	35	33
10	2429	2137
11	17	16
12	968	879
13	8	8
14	400	370
15	4	4
16	170	159
17	2	2
18	73	69
19	1	1
20	32	31
22	14	14
24	6	6
26	3	3
28	1	1
30	1	1

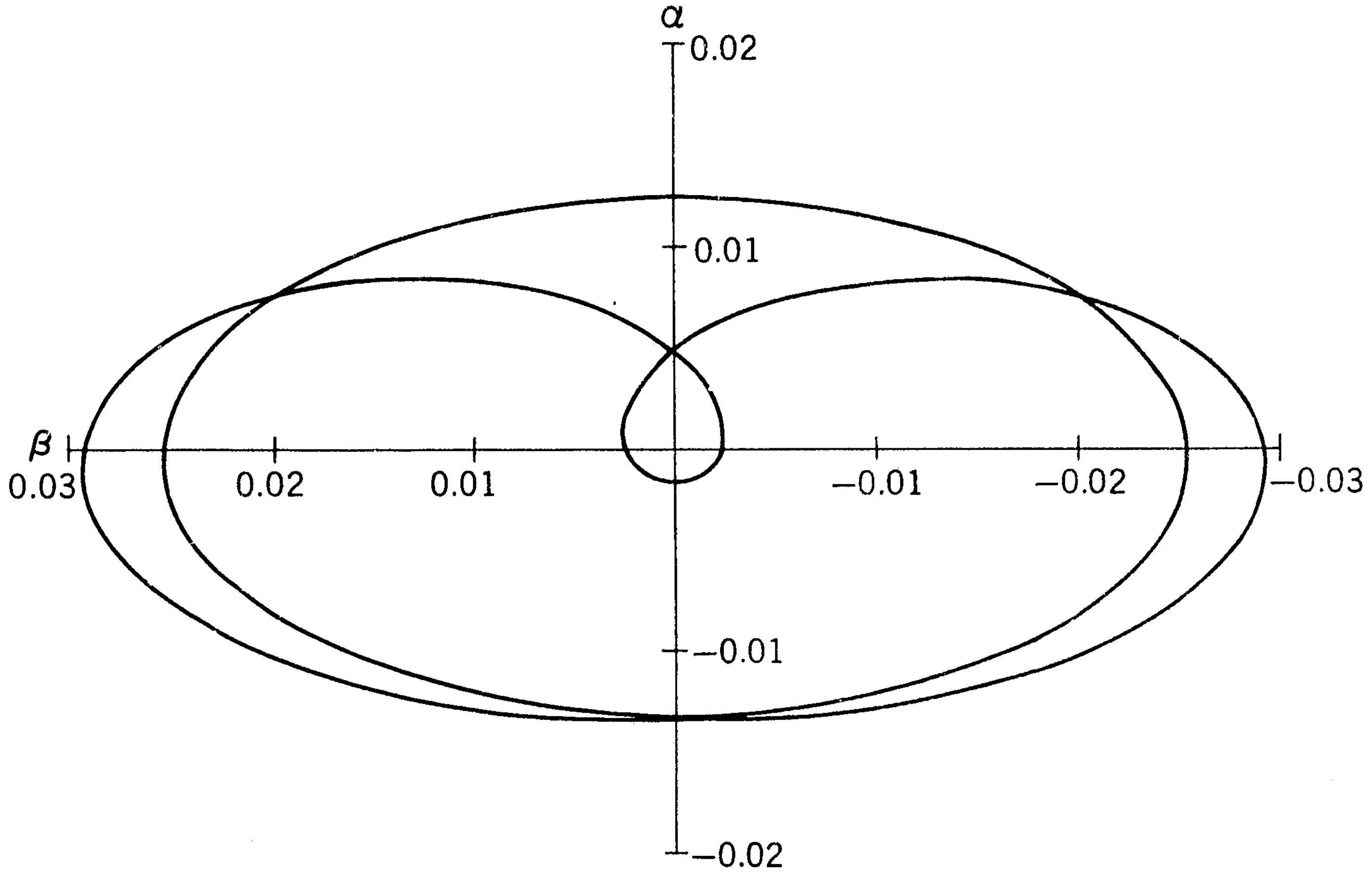

Figure 3—Deviations from near circular motion of a stable periodic orbit between the 2/1 and 3/2 resonances. $m' = 1/1047.35$, $\nu = 1.6$, $e_2 = -0.008\ 292$, $e_3 = 0.007\ 791$.

Table 3
Coefficients of the trigonometric series for the orbit of Figure 3

k	$\alpha_k \cdot 10^6$	$\beta_k \cdot 10^6$
0	-235	0
1	1842	-8567
2	7202	-17674
3	-8168	15207
4	-1234	1935
5	-522	585
6	-185	271
7	-100	128
8	-56	66
9	-32	37
10	-19	21
11	-11	13
12	-7	8
13	-4	5
14	-3	3
15	-2	2
16	-1	1
17	-1	1
18	-1	1

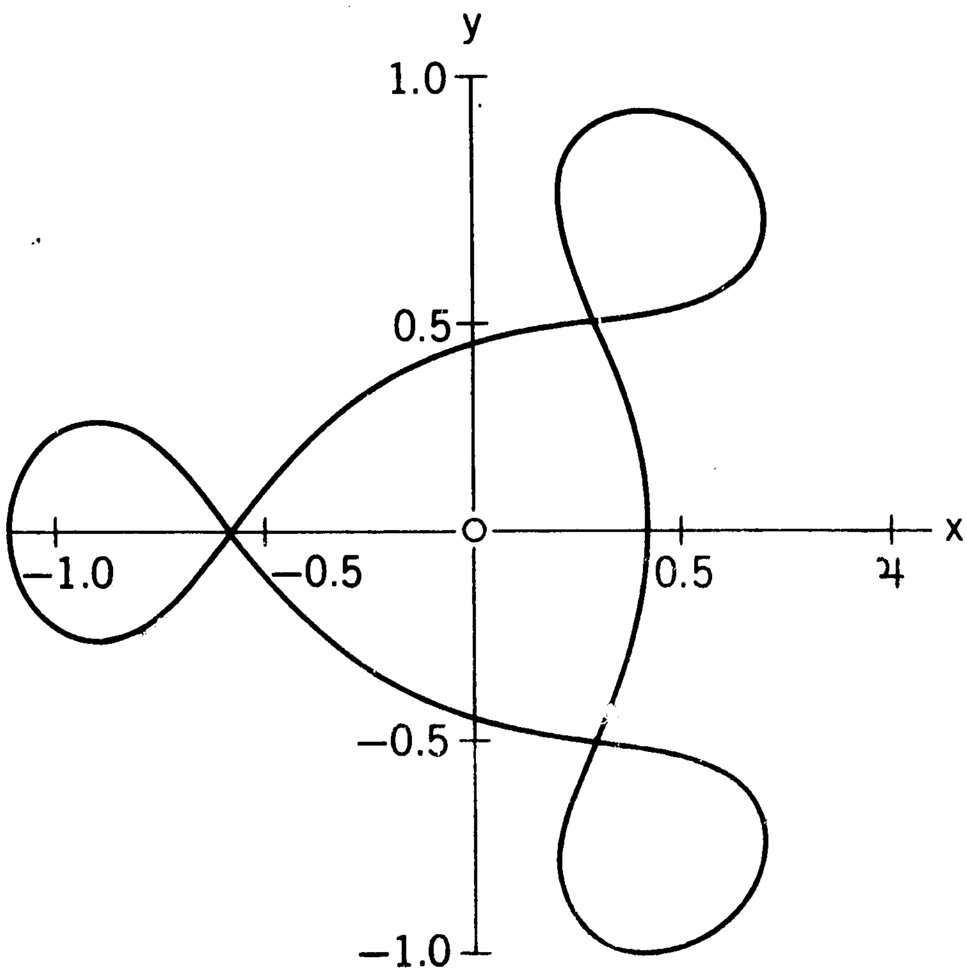

Figure 4a—Synodic motion of a stable periodic orbit at the Hilda resonance. m′ = 1/1047.35, ν = 1.5, e_3 = 0.453 692.

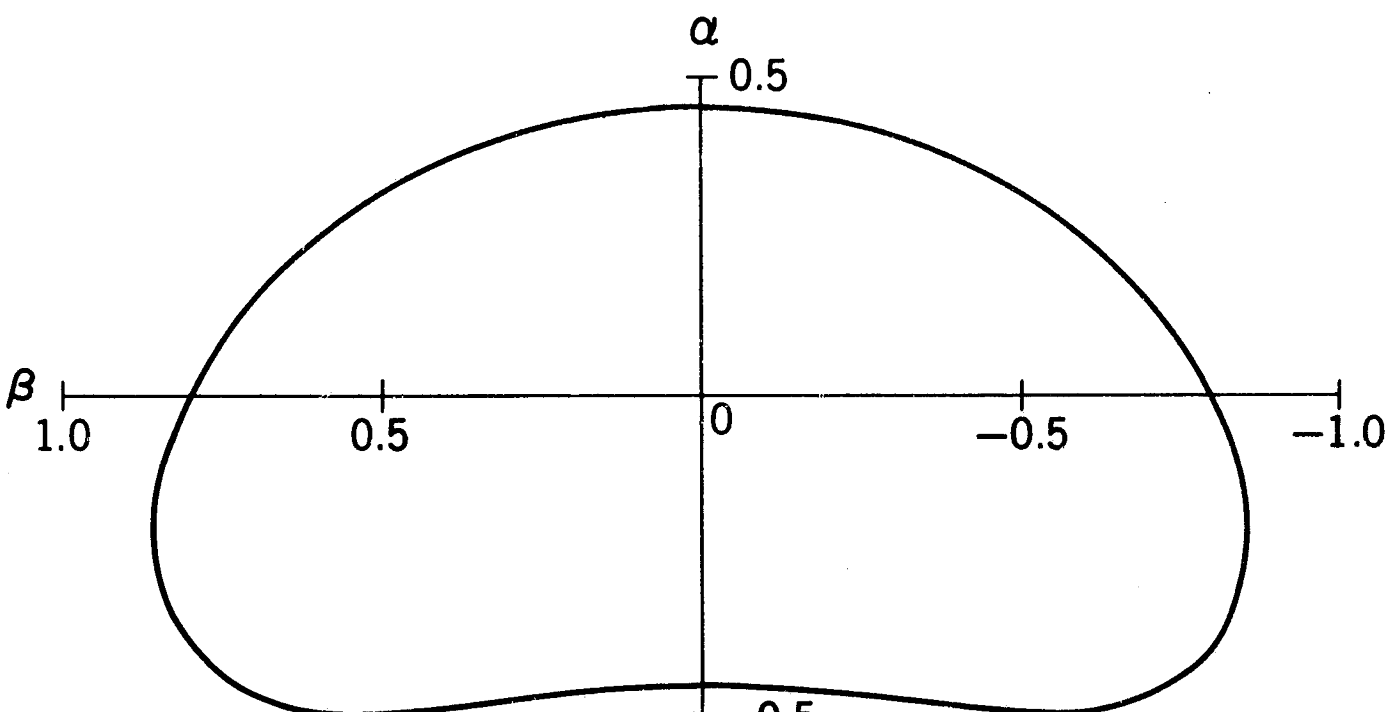

Figure 4b—Deviations from circular motion for the orbit of Figure 4a. The figure consists of three nearly identical loops similar to those of Figure 2b except that the eccentricity is now larger.

Table 4
Dominant coefficients of the trigonometric series for the orbit of Figures 4a and 4b

k	$\alpha_k \cdot 10^6$	$\beta_k \cdot 10^6$
0	-103710	0
3	-487824	873251
6	89846	32529
9	27907	19671
12	11018	8978
15	4778	4144
18	2187	1963
21	1038	952
24	506	471
27	252	236
30	127	120
33	65	62
36	34	32
39	18	17
42	9	9
45	5	5
48	3	3
51	1	1
54	1	1

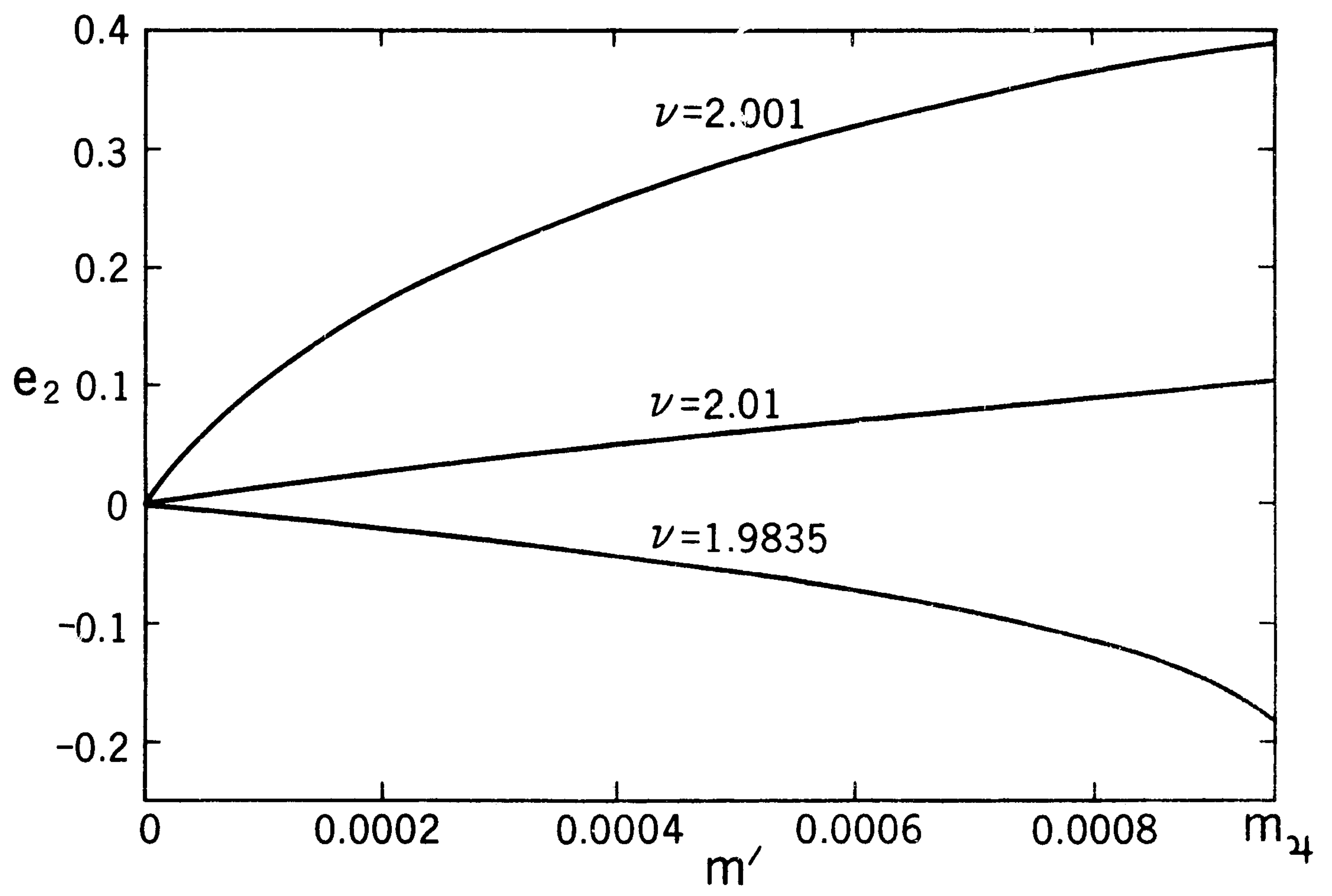

Figure 5—Three families of the first kind. These families start from circular motion at $m' = 0$ and are continued up to $m' = 1/1047.35$ holding the ratio of mean motions, $\nu = n/n'$, at a fixed value. These curves show the variations of the eccentricity e_2 for three values of ν near 2. The cases $\nu = p/(p-1)$ for integral $p > 1$ are singular for continuations of this type.

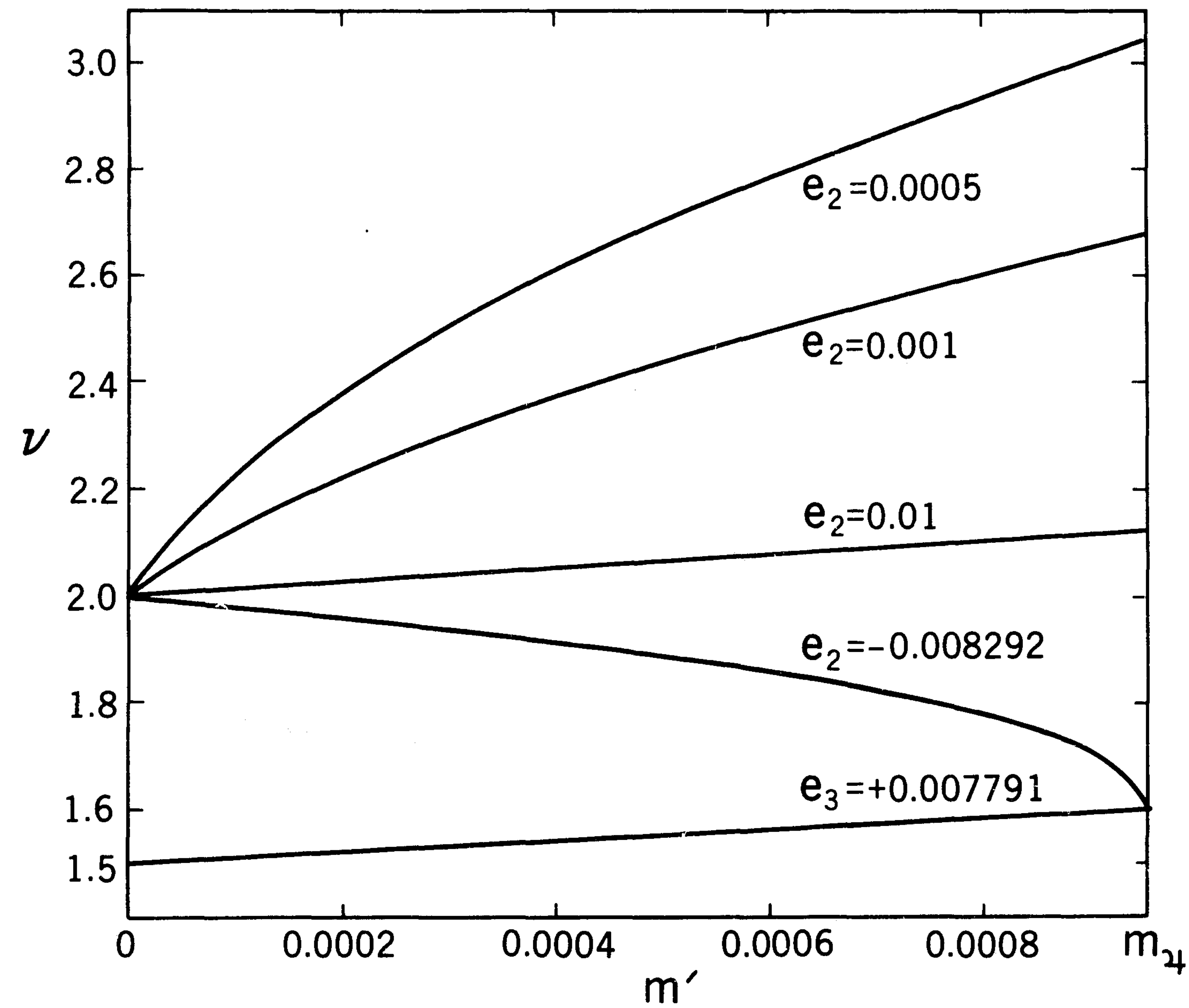

Figure 6–Five families of the second kind. These families start from elliptic motion at exact resonance $\nu = p/(p-1)$ for $m' = 0$ and are continued up to $m' = 1/1047.35$ holding the eccentricity e_p fixed. The ratio of mean motions, ν, is computed as part of the solution. Curves are shown for four families starting at $\nu = 2$ and one family starting at $\nu = 3/2$. The families $e_2 = -0.008\,292$ and $e_3 = 0.007\,791$ intersect at the orbit of Figure 3. The cases $e_p = 0$ are singular for continuations of this type from $\nu = p/(p-1)$.

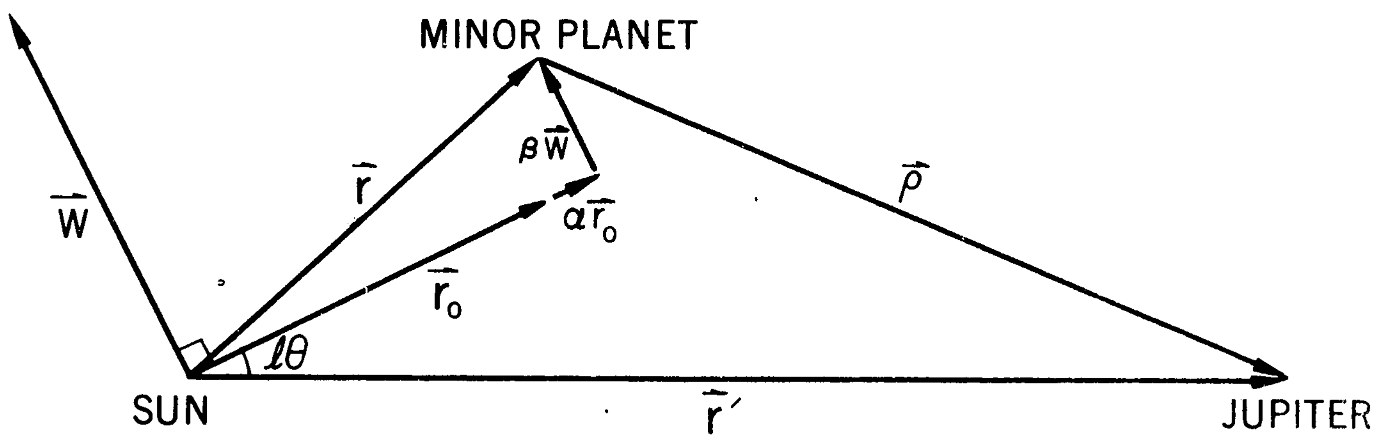

Figure 7–Geometry of the coordinate systems.

$|\mathbf{r}_0| = a, \quad |\mathbf{w}| = a, \quad |\mathbf{r}| = a\,[(1+\alpha)^2 + \beta^2]^{1/2}, \quad |\mathbf{r}'| = a', \quad \ell\,\theta = (n - n')t,$

$$|\boldsymbol{\rho}| = a\left\{(1+\alpha)^2 + \beta^2 + \left(\frac{a'}{a}\right)^2 - 2\left(\frac{a'}{a}\right)\left[(1+\alpha)\cos \ell\,\theta - \beta \sin \ell\,\theta\right]\right\}^{1/2}$$

CONTROLS ON THE COMPUTATION

The Jacobi constant plays no role in the computational scheme used in this study. Its value is monitored as an indicator of numerical inaccuracies or difficulties.

The iteration process is continued until the residuals in the differential equations fall below an acceptable tolerance. The number of terms kept in the series and the order of the matrix used in the iteration are determined for each case so as to give the required accuracy and convergence. Generally it was required that the last ten terms computed in the series be less than $0.5 \cdot 10^{-12}$.

When the iteration matrix becomes ill-conditioned, a change is made to a different independent parameter. No cases have been encountered where this presents any serious difficulty.

Several cases have been checked by comparison with numerical integration, and the expressions for the variations have been checked by numerical differencing of the series coefficients.

COMPARISON WITH NUMERICAL INTEGRATION

The author is aware of no numerical integration program which has the facility for continuing families of orbits in all of the ways discussed here, or which gives the variations with respect to a parameter of a representation of the entire orbit such as those given in equations (E) and (F). On the other hand, the present technique is not suitable when the series convergence becomes too slow, nor is it useful for computing orbits which are not periodic. The necessary modifications for regularization have not yet been made.

It should be mentioned that, although the restricted problem of three bodies is extremely rich in natural families (Wintner) of periodic orbits, elements of many of these families have been discovered only by accident. Continuation from two-body motion as suggested by Poincaré and applied here provides a basis for the systematic exploration of many of these families of orbits.

The present method can be used in conjunction with a good numerical integration program taking full advantage of the special features of each.

CONCLUSION

The present study demonstrates the strength of a semi-analytic development in cases where small parameter methods fail.

The method described here has been justified mathematically for isolated periodic solutions (see Urabe (1965) and Stokes (1969)), but further work is needed for singular solutions (periodic orbits which are members of a natural family) as in this study. The indeterminancies are removed here by applying constraints. The value of a suitable parameter is specified, and the epoch of time is defined to occur at a perpendicular crossing of the axis for symmetric orbits. For non-symmetric orbits one might require one of the leading coefficients in the series to be zero to tie down the epoch.

In the restricted problem of four-bodies, once the motions of the three primaries are determined, the periodic solutions for the infinitesimal body are isolated, so these difficulties are not encountered. Nor would they be present in the reduced (or elliptic) problem of three-bodies.

There are many possible applications of this method in celestial mechanics problems, and several are being explored.

ACKNOWLEDGMENTS

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